Technical Comments

Comment on "Dynamic Characteristics of Rotor Blades: Integrating Matrix Method"

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A RECENT paper by Murthy[1] presented an analysis of the coupled flapwise bending-torsion vibration characteristics of rotor blades using the integrating matrix method. Murthy has drawn several incorrect conclusions relative to the general applicability of this method; furthermore, the eigenvectors calculated using the integrating matrix method are in error. The integrating matrix method, however, is well established in the literature as having a wide range of applicability for obtaining numerical solutions of different equations in structural mechanics.

Murthy stated that the coupled flapwise bending-torsion equations are those derived by Houbolt and Brooks.[2] In fact, the equations considered by Ref. 1 are a subset of the corresponding flapwise bending-torsion equations of Ref. 2. The following additional assumptions are inherent in the equations considered by Ref. 1: a) the blade was assumed to have zero twist, b) the tension axis was assumed to coincide with the elastic axis, c) the component of tensile stress in a plane normal to the elastic axis was assumed to produce a negligible torsional resisting moment, and d) the blade was assumed to have zero root offset from the axis of rotation. Murthy's solution of the governing equations was apparently taken directly from Appendix A of Ref. 3. This appendix was solely intended as a specialized application of the integrating matrix operator developed by Hunter.[4] However, Murthy incorrectly concluded that the method has a limited range of applicability. Specifically, Murthy concluded that the integrating matrix method was only applicable to equal length segments. In fact, the method was easily generalized by Vakhitov[5] to include arbitrary integration intervals for nonuniform beams. Murthy further concluded that the solution of the coupled bending-torsion equations required the introduction of a differentiating matrix operator. However, while a differentiating matrix operator is a feasible approach, it is neither necessary nor desirable.

The governing equations of Ref. 1 can be expressed in the corresponding form used in Ref. 6 as

$$EIw''(x,t)+\Omega^2\int_x^R \{m\eta[w(\eta,t)-w(x,t)]+me\eta\phi(\eta,t)\}\mathrm{d}\eta$$

$$=-\int_x^R [m\ddot{w}(\eta,t)+me\ddot{\phi}(\eta,t)](\eta-x)\mathrm{d}\eta \tag{1}$$

$$GJ\phi'(x,t)+\Omega^2\int_x^R [m(k_{m_2}^2-k_{m_1}^2)\phi(\eta,t)+me\eta w'(\eta,t)]\mathrm{d}\eta$$

$$=-\int_x^R [me\ddot{w}(\eta,t)+mk_m^2\ddot{\phi}(\eta,t)]\mathrm{d}\eta \tag{2}$$

$$T=\Omega^2\int_x^R m\eta\mathrm{d}\eta \tag{3}$$

Formal differentiation of these equations yields the corresponding set considered in Ref. 1. Assuming simple harmonic motion and applying the intergrating matrix operator of Ref. 4 to the preceding equations yields

$$[A]\{\alpha\}=\lambda[B]\{\alpha\} \tag{4}$$

where the elements $[A]$ and $[B]$ are given in Ref. 6. Here the eigenvector contains curvatures in bending and slopes in torsion, $\{\alpha\}=\{w_1'', w_2''...w_n'',\phi_1',\phi_2'...\phi_n'\}$; in contrast, the eigenvectors of Ref. 1 contain bending and torsional displacements, $\{\alpha\}=\{w_1,w_2...w_n,\phi_1,\phi_2...\phi_n\}$. The eigenvectors of Eq. (4) can be numerically integrated using the integrating matrix operator to yield the coupled bending-torsion displacements. Thus, the introduction of a differentiating matrix operator is unnecessary.

Equation (4) was solved with fixed-free boundary conditions using 11 stations and the seventh-degree integrating matrix operator derived in Ref. 4. Table 1 compares the first three coupled mode shapes calculated from Eq. (4) with the transmission matrix results in Ref. 1. Table 1 shows excellent agreement between the integrating matrix and transmission matrix methods. However, Ref. 1 found significant differences in the nonpredominant components of the coupled modes and attributed these differences to the nonsymmetric form of $[A]$ and $[B]$. Indeed, many numerical techniques yield nonsymmetric matrices. The nonsymmetrix form of $[A]$ and $[B]$ in Ref. 1 does not support the assertions that the integrating matrix method is inaccurate or yields modes that lack orthogonality properties in the usual sense.[4]

Finally, Murthy concluded that the integrating matrix method is not applicable to a system containing rigid body degrees of freedom. In fact, Vakhitov[7] has applied an integrating matrix method to articulated rotor systems, including the limiting case of pinned-free boundary conditions.

Table 1 Comparisons of coupled flapwise bending and torsion mode shapes, fixed-free

	Integrating matrix, Eq. (4)		Transmission matrix[1]	
x/R	w	ϕ	w	ϕ
	$\omega_1=31.05$ rad/s			
0.0	0.0000	0.0000	0.0000	0.0000
0.2	0.0639	0.0014	0.0639	0.0014
0.4	0.2299	0.0026	0.2299	0.0026
0.6	0.4611	0.0038	0.4611	0.0038
0.8	0.7255	0.0046	0.7255	0.0046
1.0	1.0000	0.0049	1.0000	0.0049
	$\omega_2=193.70$ rad/s			
0.0	0.0000	0.0000	0.0000	0.0000
0.2	−0.2950	−0.0337	−0.2950	−0.0337
0.4	−0.6678	−0.0583	−0.6678	−0.0583
0.6	−0.5710	−0.0646	−0.5710	−0.0646
0.8	0.0818	−0.0550	0.0817	−0.0550
1.0	1.0000	−0.0465	1.0000	−0.0465
	$\omega_3=390.79$ rad/s			
0.0	0.0000	0.0000	0.0000	0.0000
0.2	−0.0361	0.3093	−0.0361	0.3093
0.4	−0.1242	0.5860	−0.1242	0.5860
0.6	−0.1946	0.8057	−0.1946	0.8057
0.8	−0.1963	0.9493	−0.1962	0.9493
1.0	−0.1541	1.0000	−0.1541	1.0000

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The integrating matrix method has been used successfully for many years. In contrast to the conclusions of Ref. 1, the method yields accurate results and has a wide range of applicability for obtaining numerical solutions of differential equations in structural mechanics. Thus, the assertions in Ref. 1 relative to the limitations of the integrating matrix method are without foundation.

References

[1]Murthy, V. R., "Dynamic Characteristics of Rotor Blades: Integrating Matrix Method," *AIAA Journal*, Vol. 15, April 1977, pp. 595-597.

[2]Houbolt, J. C. and Brooks, G. W., "Differential Equations of Motion for Combined Flapwise Bending, Chordwise Bending, and Torsion of Twisted Nonuniform Rotor Blades," NACA Rept. 1346, 1958.

[3]White, W. F. Jr., "Effect of Compressibility on Three-Dimensional Helicopter Rotor Blade Flutter," Ph.D. Dissertation, Aerospace Engineering Dept., Georgia Institute of Technology, Atlanta, Ga., 1973.

[4]Hunter, W. F., "The Integrating Matrix Method for Determining the Natural Vibration Characteristics of Propeller Blades," NASA TN D-6064, Dec. 1970.

[5]Vakhitov, M. B., "Integrating Matrices as a Means of Numerical Solution of Differential Equations in Structural Mechanics," *Izvestiya VUZ. Aviatsionnaya Tekhnika*, No. 3, 1966, pp. 50-61.

[6]White, W. F. Jr. and Malatino, R. E., "A Numerical Method for Determining the Natural Vibration Characteristics of Rotating Nonuniform Cantilever Blades," NASA TM X-72, 751, Oct. 1975.

[7]Vakhitov, M. B., "Calculation of Free, Coupled Bending-Torsion Oscillations of Rotating Helicopter Blades," *Izvestiya VUZ. Aviatsionnaya Tekhnika*, No. 4, 1963, pp. 47-69.

Reply by Author to W. F. White Jr.

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THE author greatly appreciates the interest of William F. White Jr. in this work (Ref. 1). It appears that most of the comments arose due to the lack of precise statements in the Technical Note. There was a comment regarding the way of referencing the equations of motion [Eqs. (1) and (2) of Ref. 1] to the original work (Ref. 2). The author feels that it is a perfectly legitimate, even though not precise way of giving the reference. For instance, White stated that the elements of $[A]$ and $[B]$ of Eq. (4) of the Technical Comment are given in Ref. 3. In fact, the elements of $[A]$ and $[B]$ are a subset of those given in Ref. 3. Also, it was stated that the formal differentiation of Eqs. (1-3) yields Eqs. (1-3) of Ref. 1. In fact, Eqs. (1-3) of Ref. 1 were obtained for a simple harmonic motion which is a special case of Eqs. (1-3) of the Technical Comment. These statements are not highly precise, but are always understandable and, in the opinion of the author, are not worthy of comment. The following additional conditions are inherent in Eqs. (1-3) of the Technical Comment

$$EIw''(R,t)=0; \quad GJ\phi'(R,t)=0; \quad T(R)=0$$

Eqs. (1) to (3) of Ref. 1 are independent of similar conditions on the curvature and slope curves of amplitudes of simple harmonic flapwise deflection and twisting deformation, respectively, and similarly independent of the condition on the tension curve. These conditions enter through the boundary conditions of the system. It was not the intention or objective of Ref. 1 to list or discuss the various assumptions and limitations of the equations of motion derived in Ref. 2, and several papers exist in the literature for that purpose. The main objectives of the Technical Notes were: 1) to investigate the applicability of the integrating matrix technique (using exclusively the integrating matrices and their functions) to compute the natural frequencies and natural mode shapes of a system containing rigid body degrees of freedom whose eigenvalue problem is described by the differential equations and the associated set of necessary and sufficient boundary conditions, and 2) to determine the accuracy of coupled mode shapes yielded by this technique when this is applied to the differential equation formulation of the system containing no rigid body degree of freedom.

These objectives were motivated by the following aspects of this technique: 1) it gives rise to a flexibility type of formulation, and 2) it yields a nonself-adjoint formulation. It was clearly shown in Ref. 1 that the integrating matrix technique is not applicable to a system containing rigid body degrees of freedom by taking an example of a uniform continuous nonrotating beam with flapwise bending degree of freedom which is hinged at one end and free at the other end. The author would like to offer a word of caution here—that the problem under consideration is a system containing a rigid body degree(s) of freedom whose eigenvalue value problem is described by the differential equation(s) and the associated set of complete boundary conditions and for which the numerical solution is sought by the exclusive use of the integrating matrices and/or their functions. Any published analysis of such systems whose eigenvalue problem is described by integro-differential equations of the type considered in Ref. 3 would add to the utility of this technique. It was never specifically concluded in the Technical Note that the integrating matrix technique is applicable only to equal increments of the independent variable and the problem is merely defined and formulated for equal intervals to make use of the more readily available results[4] to achieve the specified objectives. It is obvious that as long as the numerical interpolation formulas are feasible for unequal intervals, then the technique is feasible for those intervals. Not all of the systems whose equations of motion are described by differential equations can be analyzed by the exclusive use of integrating matrices. The introduction of differentiating matrices would be desirable if one is looking for the advantage of having a technique which uses merely the coefficients of the differential equations of motion and/or if one is looking for a numerical technique which can solve the equations at hand rather than converting the equations to a different form for the sake of a numerical technique. There are several numerical techniques available in the literature to suit a given form of equations of motion.

The integrating matrix results presented in Ref. 1 are not in error, but an error was made in directly comparing these results with those obtained by the transmission matrix method. The mode shapes obtained by these techniques were normalized in different ways. Hence, the conclusion of Ref. 1 that the integrating matrix technique does not yield accurate deflections for nonpredominant modes in a coupled mode shape is erroneous. The author regrets the error and is thankful to White for pointing it out.

The following are to be added to Ref. 1 for completeness. The numerical value for the mass is missing for the results presented in Table 1 ($m=0.0015$ slugs/in.). The matrices $[EI]$, $[m]$, $[me]$, $[GJ]$, $[x]$, $[mk_m^2]$, and $[m(k_{m2}^2-k_{m1}^2)]$ are diagonal matrices. The matrix $[I]$ is the integrating matrix defined in Ref. 4. The definitions of $[F]$ and $[P]$ matrices are as follows:

$$[P]=\text{diag}([F]\,[m]\,[x]\,\{1\})$$

$$[F]=([D_n]-[I])$$

$[I]$ = unit matrix, $[D_n]$ is defined in Ref. 4

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